ENGINEERING MATH SM314
PRACTICE FINAL EXAM, 2003

You must show all of your work to receive credit!
Point distribution:

- Laplace Transforms 40
- Probability & Statistics 100
- Complex number's & Taylor Series 30
- Fourier Analysis 30

Total points: 200

LAPLACE TRANFORMS: (Do both)

1. (a) Use Laplace transforms to solve the initial value problem (IVP)

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 0, x(0) = 1, x'(0) = -2.$$

(b) Given that the Green's function for the system above is $G(t) = e^{-t}\sin(t)$, write the solution to $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = f(t), x(0) = 0, x'(0) = 0$ as a convolution integral.

(c) Use the superposition principal to show how you would use the answers to (a) and (b) to
solve $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = f(t), x(0) = 1, x'(0) = -2.$

2. Given the IVP $(D+2)y(t) = f(t)$, for $t > 0,\ x(0) = 0$.
Find the
(a) the transfer function;
(b) the Green's function, i.e., the response to the Dirac delta function;
(c) Express the response of the system to an arbitray input, $f(t)$, by means of a convolution integral using the Green's function found in (b).
(d) Use (c) to find the response to $f(t) = e^{-t}U(t-1)$.

PROBABILITY & STATISTICS: (Do 5 out of 6)

1. Let the discrete random variable (abb. r.v.) X have cumulative distribution function (CDF)

$$F(x) = Prob(X \leq x) = \begin{cases} 0 & if\ x < 1 \\ .1 & if\ 1 \leq x < 3 \\ .4 & if\ \ 3 \leq x < 5 \quad x \\ .9 & if\ 5 \leq x < 5.5 \\ 1 & if\ \ x \geq 5.5 \end{cases}$$

(a) Find the probability mass function of X, $p(x) = Prob(X = x)$.
(b) Find $Prob(3 \leq X \leq 5.5)$, $Prob(X < 4)$ and $Prob(X = 3.5)$.
(c) Find E(X) and Var(X).

2. The life time of a cathode ray tube in hours is a is a r.v. T with CDF $F(t) = Prob(T \leq t) = \begin{cases} 0 & if \ \ t < 0 \\ 1 - e^{-1000t} & if \ \ t \geq 0 \end{cases}$

(a) Is T a discrete or continuous r.v.? Why?

(b) For T find its probability function f(t).

(c) Find the probability that such a tube will be operating for:

(i) at least 500 hr.'s, (ii) less than 1000 hr.'s, (iii) between 500 and 700 yr.'s.

3. A large lot of fuses has a certain proportion $p(0 < p < 1)$ which are defective. If 100 of these fuses are randomly selected, find the probability that

(a) all fuses are defective.

(b) at least one fuse is not defective.

(c) if p = .001 find the probability that not more than 5 fuses are defective.

(HINT: This is a binomial problem. Answers to (a) and (b) should be in terms of p.)

4. Service times for customers coming through a checkout counter in a retail store are independent r.v.'s with a mean 1.5 minutes and a variance of 1.0. Let T be the time min.'s it takes to serve 100 customers. (Note the actual distribution of the individual service times is unknown.

(a) Briefly explain why T can be assumed to have a normal distribution.

(b) Find mean and variance of T.

(c) Find the probability that the 100 customers can be served in 2 3/4 hr.'s.

(d) Find a t $_0$ so that 96.06% of the time it takes at least t_0 minutes to serve the 100 customers.

5. Suppose that the joint continuous r.v. [X,Y] has probability density function

$$f(x, y) = \begin{cases} kx^2y^2 & if \quad -1 < x < 1, -1 < y < 1 \\ 0 & elsewhere \end{cases}$$

(a) Find k.

(b) Find the marginal PDF's $f_X(x)$ and $f_Y(y)$.

6. An experiment was conducted to estimate the average weight loss of grossly overweight men on a special 2-week diet. A sample of 50 men had an average weight loss of 9.44 lbs with a standard deviation of 3.68.

(a) Find a 95% two-sided confidence interval for the true mean weight loss.

(b) Find a 95% one-sided upper confidence interval for the mean weight loss.

(c) How many such men would have to be sampled to 99% confident that the error incurred in using the average to approximate the true mean weight loss is less than 2 lbs.

COMPLEX NUMBERS & TAYLOR SERIES: (Do both)

1. (a) Find all roots of 5th roots of -32, $(-32)^{1/5}$, and write these in rectangular form, i.e., $x + iy$.

(b) Find all square roots of -i. i.e., $(-i)^{1/2}$, and write these in rectangular form, i.e., $x + iy$.

(c) Explain how you can tell, without actually finding them, whether the roots in (a) and must occur in

complex conjugate pairs.

(d) Find Re $\left[\frac{1+i}{1-i}\right]$

(e) Find a value of k which makes z=2 a root of $z^3 + kz^2 - z + 1 = 0$. For this value of k show that $z = 2$ cannot be a double root.

2.(a) Using known series, find the Taylor series about x = 0, i.e., the MacLaurin series for $f(x) = \frac{x}{1-x^2}$. Also state the interval of convergence of this series.

(b) Using Taylor series write out the first four (4) nonzero terms in the Taylor series

solution of the IVP $y'' - xy = 0, y(1) = 0, y'(1) = -1$.

FOURIER ANALYSIS: (Do 3 of the 4)

1. Let

$$f(t) = \begin{cases} 1 & if \quad -\pi/2 \le t < \pi/2 \\ 0 & if \quad t \in [-\pi, -\pi/2) \quad or \ if \ t \in [\pi/2, \pi) \end{cases}$$

with $f(t + 2\pi) = f(t)$ for all t.

(a) Expand f(t) in a Complex Fourier Series (CFS). Write the result in sigma notation.

(b) Draw a graph of the function to which the CFS converges for $-3\pi \le t \le 3\pi$. Also explicitly state what the CFS converges to at $t = \pi$ and at $t = -3\pi/2$.

(c) From the result in sigma notation, given in (a), write out all terms of the CFS for n = -4 to n = 4. (Note: this does include the term when n = 0.)

(d) Sketch the amplitude and phase spectra of this function for n = -3 to 3.

2. Consider the DE $(D+1)^2y(t) = f(t)$ where $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2int}$- a Complex Fourier Series where the c_n are assumed to be known. Note this does include the constant term c_0.

(a) Find a particular solution, $y_p(t)$.

(b) Find the complementary solution, $y_c(t)$.

3. Given $f(t) = u(t + 2) - u(t - 2)$.

(a) Write the Complex Fourier Integral (CFI) representation of this function.

(b) Tell what the (CFI) converges to at t = 0 and t = -2.

(c) Use the value of the (CFI) at t = 0 to evaluate the real integral

$$\int_0^\infty \frac{\sin(2\omega)}{\omega} \, d\omega.$$

4. Use Fourier transforms to do the following for the DE:

$$(D + 3)(D + 5)y(t) = f(t), \quad -\infty < t < \infty$$

(a) find the response to the Dirac delta function (impulse response).

(b) Write the response to an arbitrary f(t) as a convolution integral.